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CONJUGATE SYSTEMS OF CURVES ON A SURFACE BOTH OF WHOSE LAPLACE TRANSFORMS ARE LINES OF CURVATURE.

By J. M. Stetson.

Introduction.

The first part of this paper is devoted to the study of the relations between the radial transformation and some of the other simple transformations of general conjugate systems, and later these results are applied to the surfaces which give the paper its title. Two conjugate systems are radial transforms of each other if the straight lines joining corresponding points are concurrent. It is found that if two conjugate systems are in this relation, all conjugate systems obtained by the transformations of Levy and Laplace from one conjugate system are radial transforms of the systems similarly obtained from the other. A geometric definition of the Levy transforms in terms of the radial transforms is a consequence of this fact. In the case of conjugate systems with equal invariants it is found that the Moutard transforms are associate to the radial transforms.

Among the conjugate systems whose Laplace transforms are lines of curvature, those with equal invariants are of fundamental importance, and are the only ones considered here. For convenience, they are called conjugate systems (G). It is found that they may be defined as the conjugate systems whose Laplace transforms invert by reciprocal radii into the Laplace transforms of another conjugate system. The fact that the transformation of conjugate systems (G) so suggested is a radial transformation and therefore capable of being expressed simply has hitherto escaped attention. The fact that the systems associate to conjugate systems (G) are conjugate systems (G) has not been given geometric expression before, but Calapso has given the equations of transformation that establish that fact. In view of the relations established in the first section between the radial, associate, and Moutard transforms, the Moutard transformations of conjugate systems (G), though previously unknown, are combinations of transformations whose existence has been previously shown.

In a recent paper Eisenhart[*] considered certain *transformations K* of conjugate systems with equal point invariants. The third part establishes the existence of transformations K which when applied to a system (G) give rise to new systems (G).

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[*] Transactions of the American Mathematical Society, vol. 15 (1914), pp. 397–430.

The rest of the paper is devoted to geometric properties of conjugate systems (G). Most of them are obtained by considering a congruence of spheres which is suggested by an analytic result in the second part of this paper. Guichard's original theorems about these systems were obtained from a consideration of the same congruence. One of the results is shown to be characteristic, affording a third definition of conjugate systems (G).

1. **Laplace transformations. Radial transformations.** The envelope of the tangent planes to a surface at points along a curve form a developable surface. Through each point of the curve will pass one ruling of the developable surface. The direction determined by this ruling is called *conjugate*[*] to the direction of the curve. Two families of curves are said to form a *conjugate system* if at each point the direction of the curve of either family is conjugate to the direction of the curve of the other family which passes through that point. Every family of curves forms part of one such conjugate system, except that on every surface there exist not more than two self-conjugate systems, called *asymptotic lines*.[†] On every surface there is a unique orthogonal conjugate system, known as the *lines of curvature*. A necessary and sufficient condition that the parametric lines of a surface form a conjugate system is that the coördinates x, y, z, satisfy a *Laplace equation*,

$$\frac{\partial^2\theta}{\partial u\partial v}=a\frac{\partial\theta}{\partial u}+b\frac{\partial\theta}{\partial v}, \tag{1}$$

where a and b are functions of u and v.[‡] The cöordinates x, y, z, may be thought of as defining the parametric conjugate system, and will be spoken of in this paper as the coördinates of the conjugate system.

Let the tangents be drawn to the curves u = constant of a parametric conjugate system S. In the congruence thus formed one family of developable surfaces will have for edges of regression the curves u = constant of S. From the definition of conjugate directions it is obvious that the other family of developables of the congruence is given by v = constant. The edges of regression of this family form the other focal sheet of the congruence. This surface is called a *Laplace transform* of S, and the notation S_1 is used for it. The parametric curves on S_1 are conjugate. The coördinates of S_1 are given by the equations[§]

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[*] Eisenhart, pp. 126, 127. A reference of this sort is to Eisenhart's Differential Geometry.

[†] Eisenhart, p. 128.

[‡] Eisenhart, pp. 127, 154.

[§] Eisenhart, pp. 403–406. The notation is different.

$$x_1 = x - \frac{1}{a}\frac{\partial x}{\partial v}, \qquad y_1 = x - \frac{1}{a}\frac{\partial y}{\partial v}, \qquad z_1 = z - \frac{1}{a}\frac{\partial z}{\partial v}. \tag{2}$$

The other focal surface of the congruence of tangents to the curves v = constant of S gives another *Laplace transform* S_{-1}, whose coördinates are given by

$$x_{-1} = x - \frac{1}{b}\frac{\partial x}{\partial u}, \qquad y_{-1} = y - \frac{1}{b}\frac{\partial y}{\partial u}, \qquad z_{-1} = z - \frac{1}{b}\frac{\partial z}{\partial u}. \tag{3}$$

The two Laplace transforms of S_1 are S and a new conjugate system S_2; those of S_{-1} are S and a new conjugate system S_{-2}. There is thus formed a *suite of Laplace*

$$\cdots S_{-m} \cdots S_{-2}, S_{-1}, S, S_1, S_2 \cdots S_n \cdots.$$

The Laplace transforms of any conjugate system S_i in the suite are the systems adjoining it, S_{i-1} and S_{i+1}. Here $S_0 = S$.

In order that the transformations

$$x' = \frac{x}{\lambda}, \qquad y' = \frac{y}{\lambda}, \qquad z' = \frac{z}{\lambda}, \tag{4}$$

where λ is a function of u and v, transform a conjugate system S whose coördinates are x, y, z, into another conjugate system S', it is necessary and sufficient that x, y, z, λ be four solutions of the same Laplace equation (1). The coördinates x', y', z' of S' will then satisfy the equation*

$$\frac{\partial^2\theta'}{\partial u \partial v} = \left(a - \frac{\partial}{\partial v}\log\lambda\right)\frac{\partial\theta'}{\partial u} + \left(b - \frac{\partial}{\partial u}\log\lambda\right)\frac{\partial\theta'}{\partial v}. \tag{5}$$

The transformation given by equations (4) under these conditions will be spoken of as a *radial transformation*, or, more explicitly, as the radial transformation determined by λ.

An equation of Laplace of the form (1) is said to have *equal invariants* if $(\partial a/\partial u) = (\partial b/\partial v)$. Equation (5) shows that all the radial transforms of S will have equal invariants when and only when S itself has equal invariants.

Let us find the Laplace transforms of S', defined by (4). The coördinates of S_1 are given by equations of the form (2). The coördinates of S_1', the Laplace transform of S', are given by equations of the form

$$x_1' = x' - \frac{1}{a - \dfrac{\partial}{\partial v}\log\lambda}\frac{\partial x'}{\partial v}$$

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* These statements can readily be verified by substitution.

$$=\frac{x}{\lambda}-\frac{\lambda\frac{\partial x}{\partial v}-x\frac{\partial\lambda}{\partial v}}{\left(a-\frac{\partial}{\partial v}\log\lambda\right)\lambda^2}=\frac{x_1}{\lambda_1},$$

where

$$\lambda_1=\lambda-\frac{1}{a}\frac{\partial\lambda}{\partial v}.$$

Observe that λ_1 is the Laplace transform of λ, as x_1 is of x. We therefore have

$$x_1'=\frac{x_1}{\lambda_1},\qquad y_1'=\frac{y_1}{\lambda_1},\qquad z_1'=\frac{z_1}{\lambda_1}.$$

Similarly

$$x_{-1}'=\frac{x_{-1}}{\lambda_{-1}},\qquad y_{-1}'=\frac{y_{-1}}{\lambda_{-1}},\qquad z_{-1}'=\frac{z_{-1}}{\lambda_{-1}}.$$

Repeated application of these formulæ gives

THEOREM I: *If S' is the radial transform of S determined by λ, then each member of the suite of Laplace of S' is obtained from the corresponding member of the suite of Laplace of S by the radial transformation determined by the corresponding member of the suite of Laplace of λ.*

2. **Transformations of Levy.** Levy* has shown that every surface $\overline{S}$ which is cut in a conjugate system by the developables of the congruence of tangents to the curves u = constant of a conjugate system S has the coördinates

$$\bar{x}=x-\frac{\mu}{\frac{\partial\mu}{\partial v}}\frac{\partial x}{\partial v},\qquad \bar{y}=y-\frac{\mu}{\frac{\partial\mu}{\partial v}}\frac{\partial y}{\partial v},\qquad \bar{z}=z-\frac{\mu}{\frac{\partial\mu}{\partial v}}\frac{\partial z}{\partial v},\tag{6}$$

where μ is a solution of the Laplace equation (1) satisfied by x, y, z. In the same way, surfaces $\overline{S}_{-1}$ which are cut in a conjugate system by the developables of the congruence of tangents to the curves v = constant of S have coördinates

$$\bar{x}_{-1}=x-\frac{\mu}{\frac{\partial\mu}{\partial u}}\frac{\partial x}{\partial u},\qquad \bar{y}_{-1}=y-\frac{\mu}{\frac{\partial\mu}{\partial u}}\frac{\partial y}{\partial u},\qquad \bar{z}_{-1}=z-\frac{\mu}{\frac{\partial\mu}{\partial u}}\frac{\partial z}{\partial u}.\tag{7}$$

If the same solution μ of (1) be used in (6) and (7), the lines joining corresponding points of $\overline{S}$ and $\overline{S}_{-1}$ are tangent to the curves v = constant of $\overline{S}$ and to the curves u = constant of $\overline{S}_{-1}$. The surfaces $\overline{S}$ and $\overline{S}_{-1}$ are therefore Laplace transforms of each other, and the notation used is justified.

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* "Sur quelques équations linéaires aux derivées partielles," Journal de l'École Polytechnique, cahier 56 (1886), p. 77. Cf. also Darboux, *Leçons*, vol. 2, chap. x.

The congruence of tangents to the curves u = constant of S is identical with the congruence of tangents to the curves v = constant of S_1. The coördinates of $\bar{S}$ must therefore be expressible in the form

$$\bar{x} = x_1 - \frac{\mu_1}{\frac{\partial \mu_1}{\partial u}} \frac{\partial x_1}{\partial u}, \qquad \bar{y} = y_1 - \frac{\mu_1}{\frac{\partial \mu_1}{\partial u}} \frac{\partial y_1}{\partial u}, \qquad \bar{z} = z_1 - \frac{\mu_1}{\frac{\partial \mu_1}{\partial u}} \frac{\partial z_1}{\partial u},$$

where x_1, y_1, z_1, are the coördinates of S_1, and μ_1 satisfies the Laplace equation of S_1. In fact, we have

$$\bar{x} = x - \frac{\mu}{\frac{\partial \mu}{\partial v}} \frac{\partial x}{\partial v} = x - \frac{1}{a}\frac{\partial y}{\partial v} - \frac{\mu - \frac{1}{a}\frac{\partial \mu}{\partial v}}{\frac{\partial \mu}{\partial v}} \frac{\partial x}{\partial v}.$$

But

$$\frac{\partial x_1}{\partial u} = \frac{1}{a}\left(\frac{\partial}{\partial u}\log a - b\right)\frac{\partial x}{\partial v}$$

and we take μ_1 defined by

$$\mu_1 = \mu - \frac{1}{a}\frac{\partial \mu}{\partial v},$$

$$\frac{\partial \mu_1}{\partial u} = \frac{1}{a}\left(\frac{\partial}{\partial u}\log a - b\right)\frac{\partial \mu}{\partial v},$$

so that μ_1 is the Laplace transform of μ. Then

$$\bar{x} = x - \frac{\mu}{\frac{\partial \mu}{\partial v}} \frac{\partial x}{\partial v} = x_1 - \frac{\mu_1}{\frac{\partial \mu_1}{\partial u}} \frac{\partial x_1}{\partial u}, \qquad \text{etc.,}$$

and by the theorem of Levy the coördinates of $\bar{S}_1$, the Laplace transform of $\bar{S}$, are given by

$$\bar{x}_1 = x_1 - \frac{\mu_1}{\frac{\partial \mu_1}{\partial v}} \frac{\partial x_1}{\partial v}, \qquad \text{etc.} \tag{8}$$

In like manner

$$\bar{x}_{-1} = x - \frac{\mu}{\frac{\partial \mu}{\partial u}} \frac{\partial x}{\partial u} = x_{-1} - \frac{\mu_{-1}}{\frac{\partial \mu_{-1}}{\partial v}} \frac{\partial x_{-1}}{\partial v}, \qquad \text{etc.,} \tag{9}$$

and

$$\bar{x}_{-2} = x_{-1} - \frac{\mu_{-1}}{\frac{\partial \mu_{-1}}{\partial u}} \frac{\partial x_{-1}}{\partial u}.$$

If we call $\bar{S}$, $\bar{S}_{-1}$ the *Levy transforms of S determined by μ*, repeated application of (6), (7), (8), (9) gives

THEOREM II: *If $\bar{S}$, $\bar{S}_{-1}$ are the Levy transforms of S determined by μ, then their nth Laplace transforms $\bar{S}_n$, $\bar{S}_{n-1}$ are the Levy transforms of S_n, the nth Laplace transform of S, determined by μ_n, the nth Laplace transform of μ.*

Consider now the conjugate systems $\bar{S}'$, $\bar{S}_{-1}'$, the Levy transforms of S' determined by $\mu' = \mu/\lambda$, which evidently is a solution of the Laplace equation (5) satisfied by the coördinates of S'. We have

$$\bar{x}' = x' - \frac{\mu'}{\frac{\partial \mu'}{\partial v}} \frac{\partial x'}{\partial v} = \frac{x}{\lambda} - \frac{\frac{\mu}{\lambda}}{\frac{\partial}{\partial v}\left(\frac{\mu}{\lambda}\right)} \frac{\partial}{\partial v}\left(\frac{x}{\lambda}\right)$$

$$= \frac{x}{\lambda} - \frac{\mu\lambda\left(\lambda\frac{\partial x}{\partial v} - x\frac{\partial\lambda}{\partial v}\right)}{\left(\lambda\frac{\partial\mu}{\partial v} - \mu\frac{\partial\lambda}{\partial v}\right)\lambda^2} = \frac{1}{\lambda - \frac{\mu}{\frac{\partial\mu}{\partial v}}\frac{\partial\lambda}{\partial v}}\left(x - \frac{\mu}{\frac{\partial\mu}{\partial v}}\frac{\partial x}{\partial v}\right)$$

$$= \frac{\bar{x}}{\bar{\lambda}},$$

where $\bar{\lambda}$ is the Levy transform of λ determined by μ. Similarly we have

$$\bar{x}_{-1}' = \frac{\bar{x}_{-1}}{\bar{\lambda}_{-1}}.$$

Repetition of these results gives

THEOREM III: *If S' is the radial transform of S determined by λ, the Levy transforms of S' determined by μ/λ are radial transforms of the Levy transforms of S determined by μ; and the solutions which determine the radial transformations are the corresponding Levy transforms of λ.*

If μ is a solution of (1), so is $1 - \mu$. The Levy transforms of $1 - \mu$ determined by μ are

$$1 - \mu - \frac{\mu}{\frac{\partial\mu}{\partial v}}\frac{\partial}{\partial v}(1 - \mu) = 1$$

and

$$1 - \mu - \frac{\mu}{\frac{\partial\mu}{\partial u}}\frac{\partial}{\partial u}(1 - \mu) = 1.$$

We have therefore as a consequence of the preceding theorem,

THEOREM IV: *Tangents at corresponding points to the curves u = constant of a surface and its radial transform determined by $1 - \mu$ intersect at the corresponding point of the Levy transform determined by μ.*

The corresponding statement with regard to the curves v = constant is true.

3. **Associate surfaces. Transformations of Moutard.** Two surfaces, S and S_0, are said to be *associate* if the planes at corresponding points are parallel, and to the asymptotic lines of either surface there corresponds a conjugate system on the other. If any correspondence is established between two surfaces, there will be in general one and only one system of corresponding curves which is conjugate on both surfaces.* This system is called the corresponding conjugate system. In the case of associate surfaces, it can be shown† that if the corresponding conjugate system is parametric, it has equal invariants and the coördinates of S_0 are given by the quadratures of the form

$$\frac{\partial x_0}{\partial u} = \lambda \frac{\partial x}{\partial u}, \qquad \frac{\partial x_0}{\partial v} = -\lambda \frac{\partial x}{\partial v}, \tag{10}$$

where λ is given by the equations

$$\frac{\partial \log \lambda}{\partial u} = -2b, \qquad \frac{\partial \log \lambda}{\partial v} = -2a. \tag{11}$$

It will be convenient to speak of the corresponding conjugate systems of S and S_0 as associate conjugate systems.

Since the Laplace equation must have equal invariants, we may write

$$a = \frac{\partial \varphi}{\partial v}, \qquad b = \frac{\partial \varphi}{\partial u}, \qquad \lambda = e^{-2\phi}. \tag{12}$$

Hence

$$\frac{\partial x_0}{\partial u} = e^{-2\phi} \frac{\partial x}{\partial u}, \qquad \frac{\partial x_0}{\partial v} = -e^{-2\phi} \frac{\partial x}{\partial v}, \tag{13}$$

and similar equations in y_0 and z_0.

If x and θ are two solutions of the equation

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial \varphi}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \varphi}{\partial u}\frac{\partial \theta}{\partial v},$$

the conditions of integrability of the following equations are satisfied:

$$\frac{\partial \xi}{\partial u} = e^{-2\phi}\left(x\frac{\partial \theta}{\partial u} - \theta\frac{\partial x}{\partial u}\right), \qquad \frac{\partial \xi}{\partial v} = -e^{-2\phi}\left(x\frac{\partial \theta}{\partial v} - \theta\frac{\partial x}{\partial v}\right). \tag{14}$$

It can readily be shown that

$$\frac{\partial^2 \xi}{\partial u \partial v} = \frac{\partial}{\partial v}(\log \theta - \varphi)\frac{\partial \xi}{\partial u} + \frac{\partial}{\partial u}(\log \theta - \varphi)\frac{\partial \xi}{\partial v}. \tag{15}$$

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* Eisenhart, p. 130.

† Eisenhart, pp. 378–380.

Hence the function ξ is a solution of a Laplace equation with equal invariants. Equations (14) define the *transformation of Moutard.* The conjugate system whose coördinates are given by ξ, η, ζ, where η and ζ are defined by analogous equations in y and z, is called the *Moutard transform* determined by the solution θ.

Equations (14) can be written

$$\frac{\partial \xi}{\partial u} = e^{-2\phi}\theta^2 \frac{\left(x\dfrac{\partial \theta}{\partial u} - \theta\dfrac{\partial x}{\partial u}\right)}{\theta^2},$$

$$\frac{\partial \xi}{\partial u} = - e^{-2\phi}\theta^2 \frac{\partial x'}{\partial u}, \tag{16}$$

where

$$x' = \frac{x}{\theta}.$$

Similarly

$$\frac{\partial \xi}{\partial v} = e^{-2\phi}\theta^2 \frac{\partial x'}{\partial v}, \tag{17}$$

and similar equations connect η and y', ζ and z'.

Since the Laplace equation, satisfied by x', y', z', is

$$\frac{\partial^2 \theta'}{\partial u \partial v} = \frac{\partial}{\partial v}(\varphi - \log\theta)\frac{\partial \theta'}{\partial u} + \frac{\partial}{\partial u}(\varphi - \log\theta)\frac{\partial \theta'}{\partial v},$$

equations (16) and (17) show that the Moutard transform is a conjugate system associate to a radial transform, and we have

THEOREM V: *The Moutard transforms of a conjugate system are conjugate systems associate to its radial transforms; and the solution θ which determines the Moutard transform determines also the radial transform to which it is associate.*

Similarly, the radial transformation determined by θ followed by a Moutard transformation determined by $1/\theta$ gives a conjugate system associate to the original conjugate system.

4. **Conjugate systems** (G). Conjugate systems whose Laplace transforms in both senses are lines of curvature have been studied by Guichard[*] and Calapso.[†] It is found that if (1) is the Laplace equation of the original conjugate system S, the fundamental quantities of the first order,[‡]

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[*] "Sur les réseaux qui, par la méthode de Laplace, se transforment des deux côtés en réseaux orthogonaux," Comptes Rendus, vol. 132, 1901, p. 249.

[†] "Un problema sui sistemi di linee fra loro coniugate e sulle relative trasformazioni di Laplace," Annali di Matematica, serie III, tomo XIII (1907), p. 203.

[‡] Eisenhart, p. 71.

E and G, of the surface S, are given by

$$E = e^{2\psi}\left(\frac{\partial\psi}{\partial u}\right)^2, \qquad G = e^{2\phi}\left(\frac{\partial\varphi}{\partial v}\right)^2, \tag{18}$$

where φ and ψ satisfy the equations

$$\frac{\partial\varphi}{\partial v} = a, \qquad \frac{\partial\psi}{\partial u} = b. \tag{19}$$

Conversely, any conjugate system for which E and G are so related to the coefficients of the Laplace equation has lines of curvature for its Laplace transforms.

The conjugate systems with equal invariants, for which $\varphi = \psi$, are found to be of fundamental importance. They will be called in this paper *conjugate systems* (G).* For a conjugate system (G) the fundamental quantities of the first order are

$$E = e^{2\phi}\left(\frac{\partial\varphi}{\partial u}\right)^2, \qquad F = e^{2\phi}\frac{\partial^2\varphi}{\partial u\partial v}, \qquad G = e^{2\phi}\left(\frac{\partial\varphi}{\partial v}\right)^2, \tag{20}$$

and the Laplace equation is

$$\frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial\varphi}{\partial v}\frac{\partial\theta}{\partial u} + \frac{\partial\varphi}{\partial u}\frac{\partial\theta}{\partial v}.\dagger \tag{21}$$

If the Laplace transforms, S_1 and S_{-1}, of any conjugate system S which satisfies the Laplace equation (1) are lines of curvature, $x_1^2+y_1^2+z_1^2$ is a solution of the Laplace equation of S_1, and $x_{-1}^2 + y_{-1}^2 + z_{-1}^2$ is a solution of the Laplace equation of S_{-1}. There must then be a solution λ of (1) whose Laplace transform λ_1 is $x_1^2 + y_1^2 + z_1^2$, and a solution μ whose Laplace transform μ_{-1} is $x_{-1}^2 + y_{-1}^2 + z_{-1}^2$.

We have

$$\lambda_1 = \lambda - \frac{1}{a}\frac{\partial\lambda}{\partial v}. \tag{22}$$

Differentiating and remembering that λ is a solution of (1), we have

$$\frac{\partial\lambda_1}{\partial u} = \frac{\dfrac{\partial a}{\partial u} - ab}{a^2}\frac{\partial\lambda}{\partial v}. \tag{23}$$

If we give λ_1, x_1, y_1, z_1 their values in terms of λ, x, y, z, the equations (22) and (23) give

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* Calapso uses the same notation in a less restricted sense.

† The function φ is not arbitrary, but must satisfy a differential equation of the fourth order as Calapso has shown.

$$\lambda = x^2 + y^2 + z^2 - \frac{1}{a^2}\left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Remembering that the numerator of the fraction is $G = e^{2\phi}(\partial\varphi/\partial v)^2$ and $a = \partial\varphi/\partial v$, we obtain

$$\lambda = x^2 + y^2 + z^2 - e^{2\phi}.$$

In like manner we obtain

$$\mu = x^2 + y^2 + z^2 - e^{2\psi}.$$

Evidently $\mu = \lambda$ only when $\varphi = \psi$; that is, when S is a conjugate system (G).

This result can be interpreted geometrically. The radial transformation,

$$x_1' = \frac{x_1}{x_1^2 + y_1^2 + z_1^2}, \quad \text{etc.},$$

is an inversion by reciprocal radii; the transform S_1' it gives is the Laplace transform of the radial transform of S, determined by λ. In the same way the transform S_{-1}' of S_{-1} by reciprocal radii is a Laplace transform of the radial transform of S, determined by μ. It is apparent that for S_1', S_{-1}' to be Laplace transforms of the same conjugate system S', λ and μ must be equal. Since the transform by reciprocal radii of a conjugate system Σ is not a conjugate system unless Σ is a system of lines of curvature, we have

THEOREM VI: *The conjugate systems* (G) *are the conjugate systems whose Laplace transforms invert by reciprocal radii into the Laplace transforms of another conjugate system.*

It is to be noticed that in an inversion by reciprocal radii, not only are S_1' and S_{-1}' the transforms of S_1 and S_{-1}, but S_1 and S_{-1} are the transforms of S_1' and S_{-1}'. In the case of conjugate systems (G), then, S' is a conjugate system whose Laplace transforms invert into the Laplace transforms of another conjugate system. We have proved

THEOREM VII: *If* S *is a conjugate system* (G), S', *its radial transform determined by the solution* $x^2 + y^2 + z^2 - e^{2\phi}$, *is also a conjugate system* (G); *and the Laplace transforms of either conjugate system transform by an inversion by reciprocal radii into the Laplace transforms of the other.*

The conjugate systems associate to a conjugate system (G) are defined by equations (10). The fundamental quantities are therefore

$$E_0 = e^{-4\phi}E = e^{-2\phi}\left(\frac{\partial\varphi}{\partial u}\right)^2, \qquad G_0 = e^{-2\phi}\left(\frac{\partial\varphi}{\partial v}\right)^2.$$

Since the Laplace equation of the associate conjugate systems is

$$\frac{\partial^2\theta}{\partial u\partial v} = -\frac{\partial\varphi\partial\theta}{\partial v\,\partial u} - \frac{\partial\varphi}{\partial u}\frac{\partial\theta}{\partial v},$$

they will be conjugate systems (G) if

$$E_0 = e^{2\omega}\left(\frac{\partial\omega}{\partial u}\right)^2, \qquad G_0 = e^{2\omega}\left(\frac{\partial\omega}{\partial v}\right)^2,$$

where

$$\frac{\partial\omega}{\partial u} = -\frac{\partial\varphi}{\partial u}, \qquad \frac{\partial\omega}{\partial v} = -\frac{\partial\varphi}{\partial v}.$$

These conditions are satisfied by $\omega = (\log c - \varphi)$. We have proved

THEOREM VIII: *The conjugate systems associate to a conjugate system* (G) *are conjugate systems* (G).

On account of Theorem V we have

THEOREM IX: *The Moutard transforms of a conjugate system* (G) *determined by* $x^2 + y^2 + z^2 - e^{2\phi}$ *are themselves conjugate systems* (G).

5. **Transformations** K. In a recent paper, Eisenhart* has considered transformations of conjugate systems with equal invariants. These he calls *transformations K*. The K-transform of a conjugate system S is the radial transform determined by λ of the Moutard transform determined by θ, where λ satisfies the equations

$$\frac{\partial\lambda}{\partial u} = -e^{-2\phi}\frac{\partial\theta}{\partial u}, \qquad \frac{\partial\lambda}{\partial v} = e^{-2\phi}\frac{\partial\theta}{\partial v}. \tag{24}$$

We recall some of Eisenhart's results. If we write

$$\rho = e^{-2\phi}, \tag{25}$$

the Laplace equation of S becomes

$$\frac{\partial^2\theta}{\partial u\partial v} + \frac{\partial}{\partial v}\log\sqrt{\rho}\,\frac{\partial\theta}{\partial u} + \frac{\partial}{\partial u}\log\sqrt{\rho}\,\frac{\partial\theta}{\partial v} = 0, \tag{26}$$

and equations (24) are

$$\frac{\partial\lambda}{\partial u} = -\rho\frac{\partial\theta}{\partial u}, \qquad \frac{\partial\lambda}{\partial v} = \rho\frac{\partial\theta}{\partial v}. \tag{27}$$

If X, Y, Z, denote the direction-cosines of the normals to the surface S, and X_1, Y_1, Z_1; X_2, Y_2, Z_2, the direction-cosines of the bisectors of the angles between the curves of the conjugate system S, the transformation

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* "Transformations of conjugate systems with equal point invariants," Transactions of the American Mathematical Society, vol. XV, no. 4, pp. 397–430.

K may be defined by equations of the form

$$x_1 - x = \frac{1}{\lambda}(aX_1 + bX_2 + wX). \tag{28}$$

Writing 2ω for the angle between the curves of the conjugate system S, the quantities a, b, w are solutions of the completely integrable set of equations

$$\begin{aligned}
\frac{\partial a}{\partial u} &= -\lambda\sqrt{E}\cos\omega + b\left(\sqrt{\frac{E}{G}}\frac{\partial\log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\right) + \frac{wD}{2\sqrt{E}\cos\omega} \\
&\qquad + \rho\theta\cos\omega\sqrt{E}, \\
\frac{\partial a}{\partial v} &= -\lambda\sqrt{G}\cos\omega - b\left(\sqrt{\frac{G}{E}}\frac{\partial\log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\right) + \frac{wD''}{2\sqrt{E}\cos\omega} \\
&\qquad - \rho\theta\cos\omega\sqrt{G}, \\
\frac{\partial b}{\partial u} &= \lambda\sqrt{E}\sin\omega - a\left(\sqrt{\frac{E}{G}}\frac{\partial\log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\right) - \frac{wD}{2\sqrt{E}\sin\omega} \\
&\qquad - \rho\theta\sin\omega\sqrt{E}, \\
\frac{\partial b}{\partial v} &= -\lambda\sqrt{G}\sin\omega + a\left(\sqrt{\frac{G}{E}}\frac{\partial\log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\right) + \frac{wD''}{2\sqrt{G}\sin\omega} \\
&\qquad - \rho\theta\sin\omega\sqrt{G}, \\
\frac{\partial w}{\partial u} &= -\frac{D}{2\sqrt{E}}\left(\frac{a}{\cos\omega} - \frac{b}{\sin\omega}\right), \\
\frac{\partial w}{\partial v} &= -\frac{D''}{2\sqrt{G}}\left(\frac{a}{\cos\omega} + \frac{b}{\sin\omega}\right).
\end{aligned} \tag{29}$$

If we write $T^2 = a^2 + b^2 + w^2$, T is a solution of the equations

$$\begin{aligned}
T\frac{\partial T}{\partial u} &= (\rho\theta - \lambda)\sqrt{E}(a\cos\omega - b\sin\omega), \\
T\frac{\partial T}{\partial v} &= -(\rho\theta - \lambda)\sqrt{G}(a\cos\omega + b\sin\omega).
\end{aligned} \tag{30}$$

The coördinates x_1, y_1, z_1, of the K-transform satisfy equations

$$\frac{\partial x_1}{\partial u} = \frac{\rho}{\lambda^2}\Big[\Big(a\frac{\partial\theta}{\partial u} + \theta\lambda\cos\omega\sqrt{E}\Big)X_1$$

$$
\begin{aligned}
&\qquad\qquad + \left(b\frac{\partial\theta}{\partial u} - \theta\lambda \sin\omega\sqrt{E}\right)X_2 + w\frac{\partial\theta}{\partial u}X\Bigg], \\
\frac{\partial x_1}{\partial v} &= -\frac{\rho}{\lambda^2}\Bigg[\left(a\frac{\partial\theta}{\partial v} + \theta\lambda\cos\omega\sqrt{G}\right)X_1 \\
&\qquad\qquad + \left(b\frac{\partial\theta}{\partial v} + \theta\lambda\sin\omega\sqrt{G}\right)X_2 + w\frac{\partial\theta}{\partial v}X\Bigg].
\end{aligned}
\tag{31}
$$

The Laplace equation satisfied by x_1, y_1, z_1, is

$$
\frac{\partial^2\theta'}{\partial u\partial v} = \frac{\partial}{\partial v}\log\frac{\lambda}{\sqrt{\rho}\theta}\frac{\partial\theta'}{\partial u} + \frac{\partial}{\partial u}\log\frac{\lambda}{\sqrt{\rho}\theta}\frac{\partial\theta'}{\partial v}. \tag{32}
$$

We wish to find transformations K which will transform a conjugate system (G) into another conjugate system (G). This will be done if the values of E_1 and G_1 obtained by squaring and adding equations like (31) are the same as those obtained from equation (32) in the same way as (20) are obtained from (21).

The value of E_1 from (31) is

$$
\begin{aligned}
E_1 &= \frac{\rho^2}{\lambda^4}\left[\left(a\frac{\partial\theta}{\partial v} + \theta\lambda\cos\omega\sqrt{E}\right)^2 + \left(b\frac{\partial\theta}{\partial u} - \theta\lambda\sin\omega\sqrt{E}\right)^2 + w^2\left(\frac{\partial\theta}{\partial u}\right)^2\right] \\
&= \frac{\rho^2}{\lambda^4}\left[(a^2+b^2+w^2)\left(\frac{\partial\theta}{\partial u}\right)^2 + 2\theta\lambda\frac{\partial\theta}{\partial u}\sqrt{E}(a\cos\omega - b\sin\omega) + \theta^2\lambda^2E\right].
\end{aligned}
\tag{33}
$$

The value of E_1 deduced from (32) is

$$
E_1 = \frac{\rho\theta^2}{\lambda^2}\left(\frac{\partial}{\partial u}\log\frac{\sqrt{\rho}\theta}{\lambda}\right)^2. \tag{34}
$$

Since $e^{-2\phi} = \rho$ and $(\partial\lambda/\partial u) = -\rho(\partial\theta/\partial u)$, this reduces to

$$
E_1 = \frac{\rho\theta^2}{\lambda^2}\left[\frac{\partial\theta}{\partial u}\left(\frac{1}{\theta} + \frac{\rho}{\lambda}\right) - \frac{\partial\varphi}{\partial u}\right]^2. \tag{35}
$$

When we equate the two values for E_1, we obtain, after simplification, the equation

$$
\frac{\partial\theta}{\partial u}[(\lambda+\rho\theta)^2 - \rho T^2] - 2\frac{\partial\varphi}{\partial u}\theta\lambda[(\lambda+\rho\theta) + \sqrt{\rho}(a\cos\omega - b\sin\omega)] = 0. \tag{36}
$$

Similarly, (31) gives for G_1 the value

$$
G_1 = \frac{\rho^2}{\lambda^4}\left[T^2\left(\frac{\partial\theta}{\partial v}\right)^2 + 2\theta\lambda\frac{\partial\theta}{\partial v}\sqrt{G}(a\cos\omega + b\sin\omega) + \theta^2\lambda^2G\right], \tag{37}
$$

and from (32) we deduce, using (27),

$$G_1 = \frac{\rho\theta^2}{\lambda^2}\left[\frac{\partial\theta}{\partial v}\left(\frac{1}{\theta} - \frac{\rho}{\lambda}\right) - \frac{\partial\varphi}{\partial v}\right]^2. \tag{38}$$

Equating these two values, we obtain the equation

$$\frac{\partial\theta}{\partial v}[(\lambda - \rho\theta)^2 - \rho T^2] - 2\frac{\partial\varphi}{\partial v}\theta\lambda[(\lambda - \rho\theta) + \sqrt{\rho}(a\cos\omega + b\sin\omega)] = 0. \tag{39}$$

Since $\cos 2\omega = F/\sqrt{EG}$, we have the relation

$$\frac{\partial^2\varphi}{\partial u\partial v} = \frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}\cos 2\omega. \tag{40}$$

If we differentiate equation (36) with respect to v, and simplify the results by the use of equations (25), (26), (27), (30), and (40), we obtain the equation

$$\begin{aligned}&\frac{\partial\theta}{\partial u}\frac{\partial\varphi}{\partial v}[(\lambda + \rho\theta)^2 + \rho T^2 + 2\sqrt{\rho}(\lambda + \rho\theta)(a\cos\omega + b\sin\omega - 2\sqrt{\rho}\theta)]\\ &\quad - \frac{\partial\theta}{\partial v}\frac{\partial\varphi}{\partial u}[(\lambda + \rho\theta) + \rho T^2 + 2\sqrt{\rho}(\lambda + \rho\theta)(a\cos\omega - b\sin\omega + 4\rho\theta\lambda)]\\ &\quad + 4\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}\sqrt{\rho}\theta\lambda(\sqrt{\rho}\theta - b\sin\omega) + 4\rho(\lambda + \rho\theta)\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v} = 0.\end{aligned} \tag{41}$$

A similar process applied to (39) gives the equation

$$\begin{aligned}&\frac{\partial\theta}{\partial u}\frac{\partial\varphi}{\partial v}[(\lambda - \rho\theta)^2 + \rho T^2 + 2\sqrt{\rho}(\lambda - \rho\theta)(a\cos\omega + b\sin\omega) - 4\rho\theta\lambda]\\ &\quad - \frac{\partial\theta}{\partial v}\frac{\partial\varphi}{\partial u}[(\lambda - \rho\theta)^2 + \rho T^2 + 2\sqrt{\rho}(\lambda - \rho\theta)(a\cos\omega - b\sin\omega + 2\sqrt{\rho}\theta)]\\ &\quad + 4\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}\sqrt{\rho}\theta\lambda(\sqrt{\rho}\theta - b\sin\omega) + 4\rho(\lambda - \rho\theta)\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v} = 0.\end{aligned} \tag{42}$$

If the values of $\partial\theta/\partial u$, $\partial\theta/\partial v$, given by (36) and (39) be substituted in (41) and (42), these equations are identically satisfied. Hence (29), (36) and (39) form a completely integrable system. We have proved

THEOREM X: *If a, b, w, θ, satisfy the completely integrable system* (29), (36), (39), *and if λ be determined by the quadratures* (27), *the transformations K determined by θ and λ will transform the given conjugate system* (G) *into another conjugate system* (G).

6. Geometric properties of systems (G). Let the point M, whose coördinates are x, y, z, lie on S, a conjugate system (G), and let M_1 and M_{-1}

be the corresponding points on S_1 and S_{-1}, the Laplace transforms of S. Then the length of $\overline{MM_1}$ is given by

$$\overline{MM_1}^2 = \Sigma(x - x_1)^2$$

$$= \sum\left[x - \left[x - \frac{\dfrac{\partial x}{\partial v}}{\dfrac{\partial \varphi}{\partial v}}\right]\right]^2 = \sum\left[\frac{\dfrac{\partial x}{\partial v}}{\dfrac{\partial \varphi}{\partial v}}\right]^2 = \frac{G}{\left(\dfrac{\partial \varphi}{\partial v}\right)^2} = e^{2\phi}.$$

Hence $\overline{MM_1} = e^{\phi}$. Similarly $\overline{MM_{-1}} = e^{\phi}$, and we have proved

THEOREM XI: *The lines joining a point on a conjugate system* (G) *with the corresponding points of its Laplace transforms are equal and have the length* e^{ϕ}.

We have seen that $x^2 + y^2 + z^2 - e^{2\phi}$ is a solution of the Laplace equation of the conjugate system (G). It is well known that when the Laplace equation of a conjugate system has a solution of the form $x^2 + y^2 + z^2 - R^2$, the line joining corresponding points on the two sheets of the envelope of the sphere whose center is at the point (x, y, z) of S and whose radius is R describes a congruence whose developables correspond to the conjugate system S.* In the case under consideration, $R = e^{\phi}$. We have just seen that the points M_1, M_{-1} lie on the sphere. The equation of the sphere is

$$(X - x)^2 + (Y - y)^2 + (Z - z)^2 - e^{2\phi} = 0. \tag{43}$$

The line joining corresponding points on the two sheets of the envelope of this sphere is given by the equations

$$(X - x)\frac{\partial x}{\partial u} + (Y - y)\frac{\partial y}{\partial u} + (Z - z)\frac{\partial z}{\partial u} + \frac{\partial \varphi}{\partial u}e^{2\phi} = 0, \tag{44}$$

$$(X - x)\frac{\partial x}{\partial v} + (Y - y)\frac{\partial y}{\partial v} + (Z - z)\frac{\partial z}{\partial v} + \frac{\partial \varphi}{\partial v}e^{2\phi} = 0. \tag{45}$$

For convenience we indicate equation (44), by $A = 0$, and (45) by $B = 0$. Equations (44) and (45), on account of (1), are the equations of the focal planes of the congruence of their intersection, which we will call the congruence g.

The point M_{-1} satisfies the equation $A = 0$; and the line MM_{-1} has direction-cosines proportional to $\partial x/\partial u$, $\partial y/\partial u$, $\partial z/\partial u$. Since MM_{-1} is a radius of the sphere (43), it follows that $A = 0$ is the equation of the tangent plane to this sphere at the point M_{-1}. Similarly, the plane $B = 0$ is tangent to the sphere at the point M_1.

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* Darboux, vol. ii, p. 324.

Since $A = 0$, $B = 0$, are the focal planes of the congruence g, the focal points F_A and F_B of the congruence are given* by the two systems of equations

$$A = 0, \qquad B = 0, \qquad A_{vv} = 0,$$

$$A = 0, \qquad B = 0, \qquad B_{uu} = 0,$$

where A_{vv} indicates the second partial derivative of A with respect to v, and B_{uu} the second partial derivative of B with respect to u.

It is found that

$$\begin{aligned} A_{vv} = \Sigma\left[\left(\frac{\partial\varphi}{\partial u}\frac{\partial^2 x}{\partial v^2}+\frac{\partial^2\varphi}{\partial u\partial v}\frac{\partial x}{\partial v}+\frac{\partial^2\varphi}{\partial v^2}\frac{\partial x}{\partial u}+\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}\frac{\partial x}{\partial v}+\left(\frac{\partial\varphi}{\partial v}\right)^2\frac{\partial x}{\partial u}\right)(X-x)\right] \\ +e^{2\phi}\left[2\frac{\partial^2\varphi}{\partial v^2}\frac{\partial\varphi}{\partial u}+2\frac{\partial\varphi}{\partial v}\frac{\partial^2\varphi}{\partial u\partial v}+4\left(\frac{\partial\varphi}{\partial v}\right)^2\frac{\partial\varphi}{\partial u}-\frac{\partial\varphi}{\partial u}\left(\frac{\partial\varphi}{\partial v}\right)^2-\frac{\partial\varphi}{\partial v}\frac{\partial^2\varphi}{\partial u\partial v}\right]=0. \end{aligned}$$

The use of $A = 0$ and $B = 0$ reduces this to the form

$$C = \Sigma\frac{\partial^2 x}{\partial v^2}(X - x) + e^{2\phi}\left[\left(\frac{\partial\varphi}{\partial v}\right)^2 + \frac{\partial^2\varphi}{\partial v^2}\right] = 0.$$

The equation $B_{uu} = 0$ in like manner may be replaced by the equation

$$D = \Sigma\frac{\partial^2 x}{\partial u^2}(X - x) + e^{2\phi}\left[\left(\frac{\partial\varphi}{\partial u}\right)^2 + \frac{\partial^2\varphi}{\partial u^2}\right] = 0.$$

We have the relation

$$\Sigma\frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial v^2} = \frac{1}{2}\frac{\partial}{\partial v}\Sigma\left(\frac{\partial x}{\partial v}\right)^2 = \frac{1}{2}\frac{\partial}{\partial v}G = e^{2\phi}\left[\frac{\partial^2\varphi}{\partial v^2} + \left(\frac{\partial\varphi}{\partial v}\right)^2\right]\frac{\partial\varphi}{\partial v}.$$

Therefore the point M_1 lies on the plane $C = 0$; it also lies on the plane $B = 0$. The focal point F_A also lies on both these planes. Therefore the line M_1F_A is given by the equations

$$(X - x)\frac{\partial x}{\partial v} + (Y - y)\frac{\partial y}{\partial v} + (Z - z)\frac{\partial z}{\partial v} + e^{2\phi}\frac{\partial\varphi}{\partial v} = 0,$$

$$(X - x)\frac{\partial^2 x}{\partial v^2} + (Y - y)\frac{\partial^2 y}{\partial v^2} + (Z - z)\frac{\partial^2 z}{\partial v^2} + e^{2\phi}\left[\frac{\partial^2\varphi}{\partial v^2} + \left(\frac{\partial\varphi}{\partial v}\right)^2\right] = 0.$$

The direction cosines of this line are proportional to the determinants of the matrix

$$\left\|\begin{matrix} \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\[2ex] \dfrac{\partial^2 x}{\partial v^2} & \dfrac{\partial^2 y}{\partial v^2} & \dfrac{\partial^2 z}{\partial v^2} \end{matrix}\right\|$$

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* Since the focal point is the point where the focal plane osculates the edge of regression of the developable surface it generates. See Eisenhart, p. 60.

But these are the direction-cosines of the osculating plane of the curve u = constant of S, and therefore the direction cosines of the normal to the surface S_1. Similar reasoning shows that the other focal point F_B lies on the normal to the surface S_{-1}.

The planes normal to a curve have as their envelope a developable surface whose rulings are called the *polar lines* of the curve. The polar line for a given point of a curve meets the osculating plane of the curve at that point at the center of the *osculating circle* of the curve; it meets the edge of regression of the developable at the center of the *osculating sphere* of the curve.*

Consider again the line defined by the system of equations $A = 0$, $B = 0$. On account of equation (1), this system is equivalent to the system $A = 0$, $A_v = 0$, where A_v is the partial derivative of A with respect to v. Or it is equivalent to $B = 0$, $B_u = 0$; where B_u is the partial derivative of B with respect to u. But $A = 0$ is the equation of the planes normal to the curves u = constant of S_{-1}; the equations $A = 0$, $A_v = 0$, give the polar lines of these curves. Similarly, $B = 0$, $B_u = 0$, give the polar lines of the curves v = constant of S_1. Again, the focal point F_A was defined by the system of equations

$$A = 0, \qquad B = 0, \qquad A_{vv} = 0,$$

which is equivalent to the system

$$A = 0, \qquad A_v = 0, \qquad A_{vv} = 0.$$

These equations define the centers of the osculating sphere of the curves u = constant of S_{-1}. In the same way, F_B is the center of the osculating sphere of the curve v = constant of S_1.

The tangent plane to S is the osculating plane of the curves v = constant of S_1 and of the curves u = constant of S_{-1}. The lines of the congruence g are the polar lines of these curves; their intersection with the osculating plane is the center of the osculating circle. Since the point of intersection is clearly equidistant from M_1 and M_{-1}, the osculating circles for the two systems of curves coincide.

The intersection of the normal to a surface with the polar line of a curve on a surface is the *center of normal curvature*† of the curve.

We collect the preceding results into

THEOREM XII: *If at each point of a conjugate system* (G) *a sphere of radius* e^{ϕ} *be drawn,*

1°. *The corresponding points of the Laplace transforms lie on the sphere.*

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[*] Eisenhart, pp. 15, 38, 65.

[†] Eisenhart, pp. 117, 118.

2°. *The lines joining corresponding points of the two sheets of the envelope of the sphere form a congruence* g, *whose developables correspond to the conjugate system* (G).

3°. *The focal planes of the congruence are the tangent planes to the sphere at the points where the tangent lines of the conjugate system* (G) *cut the sphere.*

4°. *The lines of the congruence may also be defined as the common polar lines of the curves* $v =$ *constant of* S_1 *and the curves* $u =$ *constant of* S_{-1}.

5°. *The focal point of the congruence corresponding to the variation of* u *is the center of the osculating sphere of the curves* $v =$ *constant of* S_1; *it is also the center of normal curvature of the curves* $u =$ *constant of* S_{-1}.

The focal point of the congruence corresponding to the variation of v *is the center of the osculating sphere of the curves* $u =$ *constant of* S_{-1}; *it is also the center of normal curvature of the curves* $v =$ *constant of* S_1.

Some of these results can be stated independently of the congruence. Thus we have

THEOREM XIII: *The curves* $u =$ *constant of* S_{-1} *and* $v =$ *constant of* S_1 *have common osculating circles at corresponding points. The center of the osculating sphere for either of these curves is the center of normal curvature for the other.*

7. Geometric characterization of systems (G). Guichard* has given a converse to one of the preceding results, which may be stated as follows:

Let Σ_1, Σ_2, be surfaces on which the lines of curvature correspond, and at corresponding points let one family of lines of curvature on one surface have the same osculating circles as the other family of lines of curvature on the other surface. Then Σ_1 and Σ_2 are the Laplace transforms of a conjugate system (G).

It is possible to transfer a part of the hypothesis to the conclusion and prove

THEOREM XIV: *Let* Σ_1 *and* Σ_2 *be corresponding conjugate systems, and let one family of curves of* Σ_1 *have at corresponding points the same osculating circles as the other family of* Σ_2. *Then* Σ_1 *and* Σ_2 *are lines of curvature and are the Laplace transforms of a conjugate system* (G).

Since the osculating circles are identical, the osculating planes are also identical. For convenience, suppose that the curves $u = c$ of Σ_1 and $v = k$ of Σ_2 have common osculating planes, and let M be the intersection of the tangent lines of these curves at the points (c, k) of each, called respectively M_{-1} and M_1. Consider the osculating planes corresponding to the values $u = c$, $v = k$, and $u = c$, $v = k + \delta k$. Since they are the osculating planes of neighboring points of $u = c$ on Σ_1 they ultimately intersect in the tangent to that curve. Since they are osculating planes of $v = k$ and $v = k + \delta k$ at points along the same conjugate curve $u = c$, they are tangent planes to the Laplace transform of Σ_2 at points along the curve $u = c$. The corresponding point of this transform being on the tangent to $v = k$ of Σ_2 at (c, k) is consequently M. M therefore generates one of the Laplace transforms of Σ_2. Similarly it generates the other Laplace transform of Σ_1. Since the lines MM_1 and MM_{-1} are therefore tangent to the curves $u = c$ and $v = k$ respectively at M, it follows from this and foregoing results that the parametric curves on the locus of M form a conjugate system. Hence Σ_1 and Σ_2 are the Laplace transforms of a single conjugate system. The coördinates x, y, z, of the point M therefore satisfy the equation

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* "Sur les cercles osculateurs et les sphères osculatrices aux lignes de courbure d'une surface," Comptes Rendus, 1912, vol. 154, page 678.

$$\frac{\partial^2\theta}{\partial u \partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{46}$$

where a and b are functions of u and v. The coördinates of M_1 and M_{-1}, the corresponding points of Σ_2 and Σ_1 are given by the equations

$$x_1 = x - \frac{1}{a}\frac{\partial x}{\partial v}, \qquad y_1 = y - \frac{1}{a}\frac{\partial y}{\partial v}, \qquad z_1 = z - \frac{1}{a}\frac{\partial z}{\partial v},$$

and

$$x_{-1} = x - \frac{1}{b}\frac{\partial x}{\partial u}, \qquad y_{-1} = y - \frac{1}{b}\frac{\partial y}{\partial u}, \qquad z_{-1} = z - \frac{1}{b}\frac{\partial z}{\partial u}.$$

Evidently, a necessary and sufficient condition that the curves v = constant of Σ_2 and u = constant of Σ_1 have the same osculating circles is that $MM_1 = MM_{-1}$ and that the curves mentioned have at corresponding points the same polar line. The polar line can be derived from the fact that it is the characteristic of the envelope of normal planes.*

The equation of the normal plane to the curve v = constant on Σ_2 is

$$\Sigma\left(X - x + \frac{1}{a}\frac{\partial x}{\partial v}\right)\frac{\partial x}{\partial v} = 0$$

or

$$\Sigma X\frac{\partial x}{\partial v} - \Sigma x\frac{\partial x}{\partial v} + \frac{G}{a} = 0.$$

For convenience we call the left-hand side of this equation K. The polar line is now given by the equations

$$K = 0, \qquad K_u = 0,$$

where

$$K_u = \Sigma X\left(a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v}\right) - \Sigma x\left(a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v}\right) - F + \frac{\partial G}{\partial u}\frac{1}{a} - \frac{G}{a}\frac{\partial}{\partial u}\log a.$$

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* Eisenhart, page 65.

Similarly, the polar line of the curves u = constant on Σ_1 are given by the equations

$$L = 0, \qquad L_v = 0,$$

where

$$L = \Sigma X \frac{\partial x}{\partial u} - \Sigma x \frac{\partial x}{\partial u} + \frac{E}{b}$$

and

$$L_v = \Sigma X \left(a \frac{\partial x}{\partial u} + b \frac{\partial x}{\partial v} \right) - \Sigma x \left(a \frac{\partial x}{\partial u} + b \frac{\partial x}{\partial v} \right) - F + \frac{\partial E}{\partial v} \frac{1}{b} - \frac{E}{b} \frac{\partial}{\partial v} \log b.$$

If the polar lines coincide, the planes $K_u = 0$ and $L_v = 0$ must both belong to the pencil $kK + lL = 0$. Now $K_u = 0$ can be written in the from

$$aK + bL - F + \frac{1}{a} \frac{\partial G}{\partial u} - \frac{1}{a} G \frac{\partial}{\partial u} \log a - \frac{a}{b} E - \frac{b}{a} G = 0.$$

If it is in the pencil, we must have

$$F - \frac{1}{a} \frac{\partial G}{\partial u} + \frac{1}{a} G \frac{\partial}{\partial u} \log a + \frac{a}{b} E + \frac{b}{a} G = 0. \tag{47}$$

Similarly, if $L_v = 0$ is in the pencil, we must have

$$F - \frac{1}{b} \frac{\partial E}{\partial v} + \frac{1}{b} E \frac{\partial}{\partial v} \log b + \frac{a}{b} E + \frac{b}{a} G = 0. \tag{48}$$

On account of the identities*

$$aF + bG = \frac{1}{2} \frac{\partial G}{\partial u},$$

$$aE + bF = \frac{1}{2} \frac{\partial E}{\partial v}$$

(47) and (48) become

$$\frac{1}{2a} \frac{\partial G}{\partial u} = \frac{G}{a} \frac{\partial}{\partial u} \log a + \frac{a}{b} E, \tag{49}$$

$$\frac{1}{2b} \frac{\partial E}{\partial v} = \frac{E}{b} \frac{\partial}{\partial v} \log b + \frac{b}{a} G. \tag{50}$$

The condition $MM_1 = MM_{-1}$ gives the relation

$$\frac{E}{b^2} = \frac{G}{a^2}. \tag{51}$$

In consequence of this relation, equations (49) and (50) become

$$\frac{\partial}{\partial u} \log \sqrt{G} = \frac{\partial}{\partial u} \log a + b, \tag{52}$$

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* Eisenhart, pp. 153, 154.

$$\frac{\partial}{\partial v}\log\sqrt{E} = \frac{\partial}{\partial v}\log b + a. \tag{53}$$

Hence we can put

$$a = \frac{\partial\varphi}{\partial v}, \qquad b = \frac{\partial\varphi}{\partial u}.$$

Now equations (52) and (53) give

$$G = a^2 e^{2\phi}\cdot V,$$

$$E = b^2 e^{2\phi}\cdot U,$$

where U is a function of u alone, and V of v alone. Equation (51) shows that, without loss of generality, we have $U = V = 1$, and obtain

$$E = \left(\frac{\partial\varphi}{\partial u}\right)^2 e^{2\phi}, \qquad G = \left(\frac{\partial\varphi}{\partial v}\right)^2 e^{2\phi},$$

while equation (46) becomes

$$\frac{\partial^2\theta}{\partial u\partial v} = \frac{\partial\varphi}{\partial v}\frac{\partial\theta}{\partial u} + \frac{\partial\varphi}{\partial u}\frac{\partial\theta}{\partial v}.$$

The parametric curves on S therefore form a conjugate system (G), and the theorem is proved.